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Asymptotic Estimates for the Sturm-Liouville Spectrum

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ASYMPTOTIC ESTIMATES FOR THE STURM-LIOUVILLE SPECTRUM

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February, 1961

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Abstract

It is shown that the differential equation

$$y'' + [\lambda + \Phi(x)]y = 0$$

can, under suitable conditions, be solved by assuming a solution of the form

$$y = A(x) \sin \varphi(x)$$

where

$$\varphi'(x) = \sqrt{\lambda + \Phi(x)} + \frac{1}{4} \frac{\Phi'(x)}{\lambda + \Phi(x)} \sin 2\varphi(x)$$

$$A'(x) = -A(x) \frac{\Phi'(x)}{2\lambda + 2\Phi(x)} \cos^2 \varphi(x) .$$

Use of the first equation leads, when boundary conditions are applied, to asymptotic estimates of the eigenvalues.

In particular, in the case of Hill's equation, it is shown that the instability intervals vanish faster than any inverse power of k, k being the order of the corresponding eigenvalues, when $\Phi(x)$ is an analytic function.

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Introduction

In the classical Sturm-Liouville problem one is concerned with the differential equation

$$\frac{d}{dx}\, p(x)\, \frac{dy}{dx} + (\lambda\rho(x) + g(x))y = 0$$

subject to the boundary conditions

$$\alpha y(0) + \beta y'(0) = 0$$

$$\gamma y(\pi) + \delta y'(\pi) = 0 .$$

The finite interval under consideration can be taken to be the interval $(0,\pi)$, λ is a parameter and $p(x)$, $\rho(x)$, $g(x)$ are real functions, and furthermore $p(x)$ and $\rho(x)$ are positive. α, β, γ, δ are constants such that $|\alpha| + |\beta| \neq 0$ and $|\gamma| + |\delta| \neq 0$. One is then interested in determining those values of λ for which non-trivial solutions of the differential equation can be found, which satisfy the boundary conditions.

If $p(x)$ and $\rho(x)$ are doubly differentiable, one can by means of the Liouville transformation reduce the equation to the form

$$\frac{d^2y}{dx^2} + (\lambda + \Phi(x))y = 0 .$$

Borg[1] discussed this equation and the asymptotic form of the characteristic values of λ. His chief interest was a discussion of the inverse Sturm-Liouville problem, where one is interested in reconstructing the differential equation from a knowledge of the Sturm-Liouville spectrum.

In this publication a method is developed, which leads to Borg's results in a very efficient fashion, and has the additional advantage of yielding improved error estimates. Furthermore under fairly general conditions one can easily obtain asymptotic series for large eigenvalues and the corresponding eigenfunctions. The method leads to particularly precise information in the case of Hill's equation.

As will be observed subsequently the method is intimately related to the W-K-B method, with the additional advantage of providing a rigorous foundation for the latter method.

The method

If we consider the differential equation

$$y'' + Q^2(x)y = 0$$

where Q(x) is a differentiable non-vanishing function, it seems reasonable to assume a solution of the form

$$y = A(x) \sin \varphi(x) .$$

If in addition one postulates that

$$y' = A(x)\ Q(x) \cos \varphi(x) ,$$

one can by means of the original equation show by a direct calculation that

$$\varphi'(x) = Q(x) + \frac{1}{2}\frac{Q'(x)}{Q(x)} \sin 2\varphi(x)$$

$$A'(x) = -A(x) \frac{Q'(x)}{Q(x)} \cos^2 \varphi(x) \; .$$

Similarly, if one assumes that

$$y = A(x) \cos \varphi(x)$$

$$y' = -A(x) \quad Q(x) \sin \varphi(x)$$

one obtains

$$\varphi'(x) = Q(x) - \frac{1}{2}\frac{Q'(x)}{Q(x)} \sin 2\, \varphi(x)$$

$$A'(x) = -A(x) \frac{Q'(x)}{Q(x)} \sin^2 \varphi(x) \; .$$

Evidently once $\varphi(x)$ has been found from the first of these equations, $A(x)$ can be obtained by an integration from the second equation, and it also follows that $A(x)/A(0)$ is a positive function, since it is an exponential.

The Sturm-Liouville problem

We now turn to the equation

$$y'' + [\lambda + \Phi(x)]y = 0$$

subject to the boundary conditions

$$\alpha y(0) + \beta y'(0) = 0$$

$$\gamma y(\pi) + \delta y'(\pi) = 0 ,$$

and subdivide the problem into four distinct subcases:

Ia. $\alpha = \gamma = 1$

$\beta = \delta = 0$

IIa. $\alpha = \delta = 1$

$\beta = 0$

γ arbitrary

Ib. $\beta = \delta = 1$

α, γ arbitrary

IIb. $\beta = \gamma = 1$

$\delta = 0$

α arbitrary

Problem Ia.

We now consider

$$y'' + [\lambda + \Phi(x)]y = 0$$

where $y(0) = y(\pi) = 0 ,$

and $\Phi(x)$ is a differentiable function of mean value zero, i.e.

$$\int_0^{\pi} \Phi(x)dx = 0 .$$

Under the assumption that

$$y = A(x) \sin \varphi(x)$$

$$y' = A(x) \sqrt{\lambda + \Phi(x)} \cos \varphi(x)$$

we obtain

$$\varphi'(x) = \sqrt{\lambda + \Phi(x)} + \frac{1}{4} \frac{\Phi'(x)}{\lambda + \Phi(x)} \sin 2\varphi(x)$$

where

$$\varphi(0) = 0$$

$$\varphi(\pi) = k\pi$$

and k is any sufficiently large integer. Here sufficiently large means so large that for the corresponding value of λ

$$\lambda + \Phi(x) > 0 .$$

The differential equation can be rewritten as an integral equation

$$\varphi(x) = \int_0^x \sqrt{\lambda + \Phi(x)}\, dx + \frac{1}{4} \int_0^x \frac{\Phi'(x)}{\lambda + \Phi(x)} \sin 2\varphi(x)\, dx .$$

From this equation we see that

$$\varphi(x) = \int_0^x \sqrt{\lambda + \Phi(x)}\, dx + O(\tfrac{1}{\lambda}) ,$$

and by an iteration

$$\varphi(x) = \int_0^x \sqrt{\lambda + \Phi(x)}\, dx + \frac{1}{4} \int_0^x \frac{\Phi'(x)}{\lambda + \Phi} \sin\left(2\int_0^x \sqrt{\lambda + \Phi(x)}\, dx\right) dx$$

$$+ \; O(\frac{1}{\lambda^2}) \; .$$

We can get estimates for λ for large values of k from the boundary condition

$$\int_0^\pi \sqrt{\lambda + \Phi(x)}\, dx + \frac{1}{4} \int_0^\pi \frac{\Phi'(x)}{\lambda + \Phi(x)} \sin\left(2\int_0^x \sqrt{\lambda + \Phi(x)}\, dx\right) dx$$

$$+ \; O(\frac{1}{\lambda^2}) = k\pi \; .$$

If $\Phi'(x)$ is differentiable an integration by parts can be applied to the second integral above and one then finds that

$$\sqrt{\lambda} = k + \frac{c_1}{8k^3} + O(\frac{1}{k^4}) \; ,$$

where

$$c_1 = \frac{1}{\pi} \int_0^\pi \Phi^2(x)\, dx + \frac{\Phi'(\pi) - \Phi'(o)}{\pi} \; ,$$

or

$$\lambda = k^2 + \frac{c_1}{4k^2} + O(\frac{1}{k^3}) \; .$$

If, on the other hand, $\Phi'(x)$ is not differentiable we expand $\Phi(x)$ in a Fourier series by continuing $\Phi(x)$ outside its interval of definition $(0,\pi)$ as an even function of period 2π and mean value zero. Then

$$\Phi(x) = \sum_{1}^{\infty} a_n \cos nx ,$$

$$a_n = o(\frac{1}{n^2})$$

and we find that

$$\sqrt{\lambda} = k + \frac{a_{2k}}{4k} + \frac{1}{8\pi k^3} \int_0^{\pi} \Phi^2(x)\, dx + O(\frac{1}{k^4}) .$$

Now that an asymptotic estimate of λ is known one can find asymptotic estimates of $\varphi(x)$ and $A(x)$ and therefore $y(x)$. Then one finds that

$$y(x) = \frac{A(0) \sin \left[2kx + \frac{1}{k} \int_0^x \Phi(x)\, dx\right]}{\left[k + \Phi(x)\right]^{1/4}} \left[1 + O(\frac{1}{k^2})\right] .$$

It is interesting to compare the eigenvalues with those of the corresponding problem for $\Phi(x) \equiv 0$. Then

$$y'' + Xy = 0$$

$$y(0) = y(\pi) = 0$$

$$\sqrt{\chi} = k$$

and

$$\sqrt{\lambda} - \sqrt{\chi} = \frac{c_1}{8k^3} + O(\frac{1}{k^4})$$

for $\Phi'(x)$ differentiable. When $\Phi'(x)$ is not differentiable

$$\sqrt{\lambda} - \sqrt{\chi} = \frac{a_{2k}}{4k} + \frac{1}{8\pi k^3} \int_0^{\pi} \Phi^2(x)\, dx + O(\frac{1}{k^4}) .$$

Problem Ib.

For the equation with constant coefficients we have in this case

$$y'' + \chi y = 0$$

$$\alpha y(0) + y'(0) = 0$$

$$\gamma y(\pi) + y'(\pi) = 0$$

so that

$$\tan \sqrt{\chi}\, \pi = \frac{(\gamma - \alpha)\sqrt{\chi}}{\chi + \gamma\alpha} .$$

One can show from this transcendental equation that

$$\sqrt{x} = k + \frac{\gamma-\alpha}{\pi k} - \frac{\left(\frac{\gamma-\alpha}{\pi}\right)^2 + \frac{1}{3}\,\frac{\gamma^3-\alpha^3}{\pi}}{k^3} + 0(\frac{1}{k^4}) .$$

When we turn to the more general problem

$$y'' + [\lambda + \Phi(x)]y = 0$$

$$\alpha y(0) + y'(0) = 0$$

$$\gamma(\pi) + y(\pi) = 0$$

we assume a solution of the form

$$y = A(x) \sin (\Phi(x) + \Delta)$$

$$y' = A(x) \sqrt{\lambda + \Phi(x)} \cos (\varphi(x) + \Delta) .$$

If

$$\Delta = -\tan^{-1} \frac{\sqrt{\lambda + \Phi(0)}}{\alpha}$$

the boundary condition at $x = 0$ is satisfied. As before we find

$$\varphi'(x) = \sqrt{\lambda + \Phi(x)} + \frac{1}{4}\,\frac{\Phi'(x)}{\lambda + \Phi(x)} \sin [2\varphi + 2\Delta]$$

with the boundary conditions

$$\varphi(0) = 0$$

$$\varphi(\pi) = k\pi + \tan^{-1} \frac{\sqrt{\lambda + \Phi(0)}}{\alpha} - \tan^{-1} \frac{\sqrt{\lambda + \Phi(\pi)}}{\gamma} .$$

As in Problem Ia. we can rewrite this equation as an integral equation and obtain the asymptotic solution

$$\varphi(x) = \int_0^x \sqrt{\lambda + \Phi(x)}\, dx + \frac{1}{4} \int_0^x \frac{\Phi'(x)}{\lambda + \Phi(x)} \cdot$$

$$\sin \left[2 \int_0^x \sqrt{\lambda + \Phi(x)}\, dx + 2\Delta\right] dx + 0(\frac{1}{\lambda^2}) .$$

Applying the boundary condition at x = π we find

$$\int_0^\pi \sqrt{\lambda + \Phi(x)}\, dx + \frac{1}{4} \int_0^\pi \frac{\Phi'(x)}{\lambda + \Phi(x)} \sin\left[2 \int_0^x \sqrt{\lambda + \Phi(x)}\, dx + 2\Delta\right] dx$$

$$+ 0(\frac{1}{\lambda^2}) = k\pi + \tan^{-1} \frac{\sqrt{\lambda + \Phi(o)}}{\alpha} - \tan^{-1} \frac{\sqrt{\lambda + \Phi(\pi)}}{\gamma} .$$

For $\Phi'(x)$ differentiable we can solve this equation asymptotically and find

$$\sqrt{\lambda} = k + \frac{\gamma-\alpha}{\pi k} - \frac{\left(\frac{\gamma-\alpha}{\pi}\right)^2 + \frac{1}{3}\,\frac{\gamma^3-\alpha^3}{\pi}}{k^3}$$

$$+ \frac{c_3}{8k^3} + 0\left(\frac{1}{k^4}\right),$$

where

$$c_3 = \frac{1}{\pi}\left[\int_0^\pi \Phi^2(x)\,dx - \Phi'(\pi) + \Phi'(0) + 4\alpha\Phi(0) - 4\gamma\Phi(\pi)\right].$$

A comparison with the corresponding value of $\sqrt{X}$ shows that

$$\sqrt{\lambda} - \sqrt{X} = \frac{c_3}{8k^3} + 0\left(\frac{1}{k^4}\right).$$

When $\Phi'(x)$ is not differentiable we proceed as before and expand $\Phi(x)$ in a Fourier series. We then obtain

$$\sqrt{\lambda} - \sqrt{X} = \frac{\int_0^\pi \Phi^2(x)\,dx}{8k^3\pi} - \frac{a_{2k}}{4k} + 0\left(\frac{1}{k^4}\right).$$

Problem IIa.

The same methods apply to this problem. The results merely will therefore be stated.

The equation

$$y'' + Xy = 0$$

$$y(0) = 0$$

$$\gamma y(\pi) + y'(\pi) = 0$$

has the eigenvalues governed by

$$\tan \sqrt{X}\,\pi = \frac{-\sqrt{X}}{\gamma}$$

from which one can find that

$$\sqrt{X} = k + \frac{1}{2} + \frac{\gamma}{\pi k} - \frac{\gamma}{2\pi k^2} + \frac{\frac{\gamma}{4} - \frac{\gamma^2}{\pi} - \frac{1}{3}\gamma^3}{\pi k^3} + 0(\frac{1}{k^4}) \,.$$

For the equation

$$y'' + [\lambda + \Phi(x)]y = 0$$

$$y(0) = 0$$

$$\gamma y(\pi) + y'(\pi) = 0$$

we have

$$\varphi'(x) = \sqrt{\lambda + \Phi(x)} + \frac{1}{4}\,\frac{\Phi'(x)}{\lambda + \Phi(x)} \sin 2\varphi(x)$$

$$\varphi(0) = 0$$

$$\varphi(\pi) = k + 1 - \tan^{-1} \frac{\sqrt{\lambda + \Phi(\pi)}}{\gamma} .$$

We then find, for $\Phi'(x)$ differentiable, that

$$\sqrt{\lambda} - \sqrt{X} = \frac{1}{8k^3} c_2 + 0(\frac{1}{k^4})$$

where

$$c_2 = \frac{1}{\pi}\left\{ \int_0^\pi \Phi^2(x)dx - 4\gamma\Phi(\pi) - \Phi'(0) - \Phi'(\pi) \right\} .$$

When $\Phi'(x)$ is not differentiable

$$\sqrt{\lambda} - \sqrt{X} = \frac{\int_0^\pi \Phi^2(x)dx}{8k^3} + \frac{a_{2k+1}}{4(k + \frac{1}{2})} + 0(\frac{1}{k^4})$$

Problem IIb.

Problem IIb can be solved immediately by realizing that under the transformation

$$x \rightarrow \pi - x$$

$$\gamma \rightarrow -\alpha$$

Problem IIa goes over into IIb and one then finds for $\Phi'(x)$ differentiable

$$\sqrt{\lambda} - \sqrt{\overline{\lambda}} = \frac{1}{8k^3} c_4 + O(\frac{1}{k^4})$$

where $c_4 = \frac{1}{\pi} \left\{ \int_0^\pi \Phi^2(x)dx + 4\alpha\Phi(0) + \Phi'(0) + \Phi'(\pi) \right\}$

and for $\Phi'(x)$ not differentiable

$$\sqrt{\lambda} - \sqrt{\overline{\lambda}} = \frac{\int_0^\pi \Phi^2(x)dx}{8k^3\pi} - \frac{a_{2k+1}}{4(k+\frac{1}{2})} + O(\frac{1}{k^4}) .$$

Hill's equation

We now consider the case where $\Phi(x)$ is an even differentiable function of period π. For boundary conditions we use the following four possible sets [see reference 2].

1.) $y'(0) = y'(\frac{\pi}{2}) = 0$

2.) $y(0) = y(\frac{\pi}{2}) = 0$

3.) $y'(0) = y(\frac{\pi}{2}) = 0$

4.) $y(0) = y'(\frac{\pi}{2}) = 0$

which lead to solutions of the differential equation with the following character respectively

1.) even of period π

2.) odd of period π

3.) even of period 2π

4.) odd of period 2π.

These four cases correspond to Problems Ib, Ia, IIb, IIa considered in the previous section, respectively. There is a slight change in the sense that the interval under consideration now is the interval $(0, \frac{\pi}{2})$, whereas before it was the interval $(0,\pi)$. The only change that must be made in cases 1.) and 2.) is that

$$\varphi(\tfrac{\pi}{2}) = k\pi .$$

We then find for cases 1.) and 2.)

$$\sqrt{\lambda} = 2k + \frac{1}{64k^3\pi} \int_0^\pi \Phi^2(x)dx + 0(\frac{1}{k^4}) .$$

In cases 3.) and 4.) we have to modify the boundary condition at π to

$$\varphi(\tfrac{\pi}{2}) = (2k + 1)\tfrac{\pi}{2} .$$

Then we find

$$\sqrt{\lambda} = 2k + 1 + \frac{1}{64k^3\pi} \int_0^\pi \Phi^2(x)dx + O(\frac{1}{k^4}) .$$

In case 2.), for example, we can show easily that

$$\varphi(x) = 2kx + \psi(x)$$

where $\psi(x)$ is an odd function of period π. If we use this form of $\varphi(x)$ as a trial solution in the equation

$$\varphi'(x) = \sqrt{\lambda + \Phi(x)} + \frac{1}{4} \frac{\Phi'(x)}{\lambda + \Phi(x)} \sin 2\varphi(x)$$

$$\varphi(0) = 0$$

$$\varphi(\frac{\pi}{2}) = k\pi$$

we obtain

$$\psi'(x) = \sqrt{\lambda + \Phi(x)} - 2k + \frac{1}{4} \frac{\Phi'(x)}{\lambda + \Phi(x)} \sin \left[4kx + 2\psi(x)\right]$$

$$\psi(0) = \psi(\frac{\pi}{2}) = 0.$$

From this equation it is evident that $\psi(x)$ is an odd function, and the equation is invariant under the transformation

$$x \rightarrow x + \pi .$$

Considering the boundary conditions on $\psi(x)$, the result is evident.

We can also show that the function $A(x)$ has period π. The differential equation for $A(x)$ is

$$A'(x) = -A(x) \frac{\Phi'(x)}{2(\lambda + \Phi(x))} \cos^2 \varphi(x)$$

$$\frac{A'(x)}{A(x)} = -\frac{\Phi'(x)}{4(\lambda + \Phi(x))} \left[1 + \cos (4kx + 2\psi(x))\right].$$

The equation is invariant under the transformation

$$x \to x + \pi .$$

Furthermore

$$\ell n \frac{A(\pi)}{A(0)} = -\frac{1}{4} \int_0^\pi \frac{\Phi'(x)}{\lambda + \Phi(x)} \Big[1 + \cos 4kx \cos 2\psi(x)$$

$$- \sin 4kx \sin 2\psi(x)\Big] \, dx ,$$

where the first integral vanishes by an immediate integration and the second and third terms vanish because they represent cosine and sine Fourier coefficients of odd and even functions. Thus

$$\ell n \frac{A(\pi)}{A(0)} = 0$$

so that $A(\pi) = A(0)$. Thus the solution

$$y = A(x) \sin \varphi(x)$$

for case 2.) can be seen to be an odd function of period π directly. It is easily shown that

$$\psi(x) = \frac{1}{4k} \int_0^x \Phi(x)\, dx + O(\frac{1}{k^2})$$

and

$$\ell n \frac{A(x)}{A(0)} = 1 + O(\frac{1}{k^2})$$

so that

$$y = A(0) \sin \left[4kx + \frac{1}{4k} \int_0^x \Phi(x)\, dx \right] + O(\frac{1}{k^2}) .$$

Evidently these asymptotic estimates can be improved if necessary.

The stability intervals for Hill's equation

It is well known [2] that if the eigenvalues corresponding to solutions of periodicity π be denoted by $\lambda_0, \lambda_1, \lambda_2, \ldots$ and those of periodicity 2π be denoted by $\lambda_1', \lambda_2', \lambda_3', \ldots$ they form a sequence ordered as follows

$$\lambda_0 < \lambda_1' \leq \lambda_2' < \lambda_1 \leq \lambda_2 < \lambda_3' \leq \lambda_4' < \lambda_3 \leq \lambda_4 < \ldots$$

For all λ in the intervals (λ_0, λ_1'), (λ_2', λ_1), (λ_2, λ_3'), (λ_4', λ_3) ...

the differential equation has only bounded solutions. For λ outside these intervals one solution is unbounded. Therefore the intervals $(-\infty,\lambda_0)$, (λ_1',λ_2'), (λ_1,λ_2), (λ_3',λ_4'), ... are known as the instability intervals and they are known to become small for high eigenvalues, but little is known regarding their order of smallness. An immediate consequence of the asymptotic formulas developed in the previous section is that

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = O(\frac{1}{k^4})$$

from which we see that

$$\lambda_{k+1} - \lambda_k = O(\frac{1}{k^3}) ,$$

and similarly,

$$\lambda'_{k+1} - \lambda'_k = O(\frac{1}{k^3}) .$$

When the function $\Phi(x)$ is analytic one can give much more precise estimates for the instability intervals. For example, for case 1.) we have

$$\varphi(\frac{\pi}{2}) = \int_0^{\pi/2} \sqrt{\lambda + \Phi(x)}\, dx - \frac{1}{4} \int_0^{\pi/2} \frac{\Phi'(x)}{\lambda + \Phi(x)} \sin 2\varphi(x) dx = k\pi$$

and for case 2.)

$$\varphi(\tfrac{\pi}{2}) = \int_0^{\pi/2} \sqrt{\lambda + \Phi(x)}\, dx + \frac{1}{4} \int_0^{\pi/2} \frac{\Phi'(x)}{\lambda + \Phi(x)} \sin 2\varphi(x)\, dx = k\pi$$

When $\Phi(x)$ is analytic so is $\Phi(x)$. Therefore one can apply integration by parts to the integral

$$\int_0^{\pi/2} \frac{\Phi'(x)}{\lambda + \Phi(x)} \sin 2\varphi(x)\, dx = \int_0^{\pi/2} \cos 2\varphi(x) \frac{d}{dx} \frac{1}{2\varphi'(x)} \frac{\Phi'(x)}{\lambda + \Phi(x)}\, dx$$

$$= -\int_0^{\pi/2} \sin 2\varphi(x) \frac{d}{dx} \frac{1}{2\varphi'(x)} \frac{d}{dx} \frac{1}{2\varphi'(x)} \frac{\Phi'(x)}{\lambda + \Phi(x)}\, dx = \ldots$$

$$= \int_0^{\pi/2} \sin 2\varphi(x) \left(\frac{d}{dx} \frac{1}{2\varphi'(x)} \right)^{4N} \frac{\Phi'(x)}{\lambda + \Phi(x)}\, dx ,$$

where the integrated terms vanish by virtue of the periodicity properties of the integrand. Since

$$\varphi(x) = 0(\sqrt{\lambda}) = 0(k)$$

$$\varphi'(x) = 0(\sqrt{\lambda}) = 0(k)$$

it follows that

$$\int_0^{\pi/2} \frac{\Phi'(x)}{\lambda + \Phi(x)} \sin 2\varphi(x)\, dx = 0\left(\frac{1}{\lambda^{2N+1}}\right) = 0\left(\frac{1}{k^{4N+2}}\right)$$

for all values of N. By taking the difference of the equations for cases 1.) and 2.) we see that

$$\sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} = 0\left(\frac{1}{k^{4N+2}}\right)$$

or

$$\lambda_{k+1} - \lambda_k = 0\left(\frac{1}{k^{4N+1}}\right)$$

since N is arbitrary we can conclude that the instability internals vanish faster than any negative power of k.

A similar process shows that when $\Phi(x)$ is m times differentiable

$$\lambda_{k+1} - \lambda_k = 0\left(\frac{1}{k^{m+1}}\right) .$$

A result due to Erdélyi.

Erdélyi derived [3] some estimates for small $\Phi(x)$. These follow very simply by this method. We write $\epsilon\Phi(x)$ instead of $\Phi(x)$ and assume that ϵ is small. Then for case 2.) we obtain

$$\varphi'(x) = \sqrt{\lambda + \epsilon\Phi(x)} + \frac{1}{4}\,\frac{\epsilon\Phi'(x)}{\lambda + \epsilon\Phi(x)}\sin 2\varphi(x)$$

$$\varphi(0) = 0$$

$$\varphi(\tfrac{\pi}{2}) = k\pi .$$

We now try to find a perturbation series solution for $\varphi(x)$ of the form form $\varphi(x) = 2kx + \epsilon\psi(x) + \ldots$

and for λ we use

$$\lambda = 4k^2 + \epsilon\mu + \ldots .$$

Retaining only ϵ terms we have immediately

$$\psi'(x) = \frac{\mu + \Phi(x)}{4k} + \frac{\Phi'(x)}{16k^2} \sin 4kx .$$

Use of the boundary conditions

$$\psi(0) = \psi(\tfrac{\pi}{2}) = 0$$

and the Fourier series

$$\Phi(x) = \sum_{n=1}^{\infty} a_n \cos 2nx$$

shows that

$$\mu = \frac{a_{2k}}{2}$$

so that

$$\lambda = 4k^2 + \frac{\epsilon a_{2k}}{2} .$$

Similarly for case 1.) we find

$$\lambda = 4k^2 - \frac{\epsilon a_{2k}}{2}$$

so that

$$|\lambda_{k+1} - \lambda_k| = |\epsilon a_{2k}| .$$

It is evident from the previous development that Erdelyi's result is not correct in general, but Magnus has shown [4] that in an average sense it is correct. If

$$\gamma_1 < \gamma_2 < \gamma_3 < \cdots$$

denote the λ corresponding to even solutions of period π and

$$\sigma_1 < \sigma_2 < \sigma_3 < \cdots$$

correspond to odd solutions of period π. Magnus showed that

$$\lambda_0 + \sum_{n=1}^{\infty} \gamma_n - \sigma_n = -\sum_{n=1}^{\infty} a_{2n} .$$

A similar result holds for cases 3.) and 4.) .

A Special Case

The previous results were all derived under the assumption that $\Phi(x)$ was differentiable. One can show that the asymptotic forms developed here do not hold when $\Phi(x)$ is not differentiable. Such an example was treated in detail in reference [5]. The equation considered there is

$$y'' + \lambda^2 Q(x) y = 0$$

$$Q(x) = 1 \qquad |x| \leq 1$$

$$= a^2 \qquad 1 < |x| \leq L$$

and it is shown there that

$$\lambda_{2n} = \left[\frac{2n}{1 + a(L-1)}\right] \frac{\pi}{2} + \theta \frac{\pi}{2}$$

$$\lambda_{2n-1} = \left[\frac{2n}{1 + a(L-1)}\right] \frac{\pi}{2} + \theta \frac{\pi}{2}$$

$$\lambda'_{2n} = \left[\frac{2n-1}{1 + a(L-1)}\right] \frac{\pi}{2} + \theta \frac{\pi}{2}$$

$$\lambda'_{2n-1} = \left[\frac{2n-1}{1 + a(L-1)}\right] \frac{\pi}{2} + \theta \frac{\pi}{2} .$$

$[x]$ denotes the greatest integer less than or equal to x and the θ's must satisfy the inequality

$$0 \leq \theta < 1 .$$

Here evidently

$$\lambda_k = \frac{k \frac{\pi}{2}}{1 + a(L-1)} + O(1)$$

and the instability intervals are $O(1)$.

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